Complex Numbers - Basic Definitions

Quadratic Equations

Examples of quadratic equations:

- $2x^2 + 3x - 5 = 0$
- $x^2 - x - 6 = 0$
- $x^2 = 4$

The **roots** of an equation are the x-values that make it "work" We can find the roots of a quadratic equation either by using the quadratic formula or by factoring.

We can have 3 situations when solving quadratic equations.

Case 1: Two roots

Example: $2x^2 + 3x - 5 = 0$

By using quadratic formula we have:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-3 \pm \sqrt{9 + 40}}{4} \\ &= \frac{-3 - \sqrt{49}}{4} \text{ or } \frac{-3 + \sqrt{49}}{4} \\ &= -2.5 \text{ or } 1 \end{aligned}$$

We find 2 roots. $x = -2.5$ and $x = 1$, as expected, showing our 2 roots:
More examples of quadratic equations with 2 roots:

$x^2 = 4$ has 2 solutions, $x = -2$ and $x = 2$.

$x^2 - x - 6 = 0$ has 2 solutions, $x = -2$ and $x = 3$.

$2x^2 + 13x - 7 = 0$ has 2 solutions, $x = -7$ and $x = ½$.

Case 2: One Root

Example: $4x^2 - 12x + 9 = 0$

Notice what happens when we use the quadratic formula this time. Under the square root we get 144 − 144 = 0.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{12 \pm \sqrt{144 - 144}}{8} \\ &= \frac{12}{8} \\ &= 1.5 \end{aligned}$$

So it means we only have **one root**. We can also say that this is a **repeated root**, since we are expecting 2 roots.

Case 3: No Real Roots

Example: $x^2 - 4x + 20 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{4 \pm \sqrt{16 - 80}}{2}$$
$$= \frac{4 \pm \sqrt{-64}}{2}$$

Under the square root, we get √(-64), and we cannot have the square root of a negative number. Can we find such a root?

Summary:

A quadratic equation has degree 2 (the highest power of x is 2) and we can have either 2 real roots, one real repeated root or something that involves the square root of a negative number.

Cubic Equations:

Cubic equations are polynomials which have degree 3 (this highest power of x is 3).

In the case of a **cubic equation**, we expect (up to) 3 real solutions:

Example 1: $x^3 - 2x^2 - 5x + 6 = 0$ has solutions $x = -2$, 1 and 3.

Example 2: If $x^3 = 8$, we know the solution $x = 2$, but we expect 2 other solutions.

Imaginary Numbers:

To allow for these "hidden roots", around the year 1800, the concept of √(-1) was proposed and is now accepted as an extension of the real number system. The symbol used is $i = \sqrt{(-1)}$ and j is called an **imaginary number**.

Powers of *i*:

$(\sqrt{a})^2 = a$, for any value of a. and $i = \sqrt{(-1)}$

Using these, we can derive the following: $i^2 = (\sqrt{-1})^2 = -1$

Multiplying by *i* again gives us:

$i^3 = i^2 \cdot (i) = -i$

$i^4 = i^3(i) = -i(i) = -(-1) = 1$

$i^5 = j^4(j) = 1 \times j = j$

$i^6 = i^5(i) = i \times i = -1$

1. Express in terms of *i* $\sqrt{(-16)}$, $\sqrt{(-100)}$, $\sqrt{-100}$, $\sqrt{(-7)}$

$\sqrt{(-2)}$ $\sqrt{(-18)}$ $\sqrt{(-2 \times -18)}$

Complex Numbers:

Complex numbers have a **real part** and an **imaginary part.**

Examples:

(1) $5 + 6j$ Real part: 5, Imaginary part: $6j$ **(2)** $-3 + 7j$ Real part: -3, Imaginary part: $7j$

NOTE: We can write the complex number $2 + 5i$ as $2 + i5$. There is no difference in meaning.

Solving Equations with Complex Numbers:

EXP: $\sqrt{(-64)}$. Now we can write the solution using complex numbers, as follows:

$$x = \frac{4 \pm \sqrt{-64}}{2}$$

$$= \frac{4 \pm j\sqrt{64}}{2}$$

$$= \frac{4 \pm 8j}{2}$$

$$= 2 - 4j \text{ or } 2 + 4j$$

Equivalent Complex Numbers:

Two complex numbers $x + yi$ and $a + bi$ are equivalent if:

The real parts are equal $(x = a)$, **and** the imaginary parts are equal $(y = b)$.

Given that $3 + 2j = a + bi$, then $a = 3$ and $b = 2$.

Exercises:

1. Express in terms of *i*: $-\sqrt{-\frac{2}{5}}$

2. Simplify:

(a) $\sqrt{-2}\sqrt{-8}$

(b) $\sqrt{(-2)(-8)}$

3. $i^2 - i^6$

4. $\left(\sqrt{-2}\right)^2 + j^4$